METRIC AND TOPOLOGICAL SPACES: RE-EXAM 2024/25

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Problem 1 (7 + 8 + 10%). (a) Can some open disk of radius 4 coincide with some open disk of radius 3 ? do

(b) Can some open disk of radius 4 be entirely and properly contained ($\subsetneq$) in some open disk of radius 3 ? no??.

(c) Can a closed disk of radius 3 around a point $x_0 \in \mathcal{X}$, $\mathcal{B}_{\bar{3}}(x_0) = \{x \in \mathcal{X} \mid \text{dist}(x, x_0) \leqslant 3\}$, not coincide with the closure of the open disk $\mathcal{B}_3(x_0)$ with the same centre and radius ? (If yes, give example; if not, prove.)

Problem 2 (10%). Give an example of a space $\mathcal{X}$ with infinitely many subsets A_n, $n \in \mathbb{N}$, such that the boundary $\partial(\bigcup_{n=1}^{+\infty} A_n)$ is not contained in the union of boundaries $\bigcup_{n=1}^{+\infty} \partial A_n$.

Problem 3 (20%). If $\mathcal{X}$ is a connected space such that $\mathcal{X} \ni x, y \mid x \neq y$ and for any $z \in \mathcal{X}$, the set $\{z\}$ is closed in $\mathcal{X}$, then the number of points in $\mathcal{X}$ is infinite. (prove) contradiction

Problem 4 (5 + 2 · 10%). (*a*) Can a non-empty *open* proper subset A ($\varnothing \neq A \subsetneq \mathcal{X}$) of a space $\mathcal{X}$ be simultaneously *closed* in $\mathcal{X}$? (If yes, give example; if not, prove.) yes, please think of that one exercise why can't I remember it?

The vector space ℓ_2 of real sequences $\boldsymbol{x} = (x_n \in \mathbb{R}, n \in \mathbb{N})$ is endowed with the Euclidean metric, $(\mathrm{d}(\boldsymbol{x}, \boldsymbol{y}))^2 = \sum_{n=1}^{+\infty}(x_n - y_n)^2$.

(*b*) Is $\mathcal{B}_1(\mathbf{0})$, the open unit disk centered at $\mathbf{0} = (0, 0, \ldots)$, compact in ℓ_2 ?

(*c*) Is $\mathcal{B}_{\bar{1}}(\mathbf{0})$, the closed unit disk centered at $\mathbf{0} = (0, 0, \ldots)$, compact in ℓ_2 ?

Problem 5 (20%). Solve for $x(s)$ the integral equation,

$$x(s) = \tfrac{1}{2} \int_0^1 s \cdot t\, x(t)\, \mathrm{d}t + \tfrac{5}{6} s,$$

by consecutive approximations starting from $x_0(s) = 0$. (In the end, verify by direct substitution that the function $x(s)$ which you have found satisfies the equation.)

Date: January 30, 2025 (18:15–21:15). Good luck !